Power Laws*

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Abstract: A power law is the form taken by a remarkable number of regularities in economics, and is a relation of the type $Y = kX^{\alpha}$, where Y and X are variables of interest, and k, α is called the power law exponent, and k is a constant. Many economic laws take the form of power laws, in particular macroeconomic scaling laws, the distribution of income, wealth, size of cities and firms, and the distribution of financial variables such as returns and trading volume. The entry surveys the theoretical explanations for the occurence of power laws.

JEL: G1, L11, O15, R12.

Power laws

A power law, also known as a scaling law, is the form taken by a remarkable number of regularities or "laws" in economics, and is a relation of the type $Y = kX^{\alpha}$, where Y and X are variables of interest, and α is called the power law exponent, and k is a typically unremarkable constant.

A special type is the distributional PL, also called a Pareto law. For instance, the probability that a firm has more than x employees is proportional to $1/x^{\zeta}$, for some positive number ζ: $P(S > x) = k/x^{\zeta}$, for some k, at least in the upper tail or most of it. The exponent ζ is independent of the units in which the law is expressed. A special case is Zipf's law, which is a Pareto law with $\zeta \simeq 1$.

Understanding what gives rise to the scaling law, and explaining the precise value of the exponent (e.g., why it is equal to 1, rather than any other

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number) is a challenge that has fascinated successive generations. Schumpeter (1949) writes: "Few if any economists seem to have realized the possibilities that such invariants hold for the future of our science. In particular, nobody seems to have realized that the hunt for, and the interpretation of, invariants of this type might lay the foundations for an entirely novel type of theory." Champernowne (1953) and Simon (1955) did make great strides in Schumpeter's vision, and the quest continues.

Power laws are also of high interest outside of economics. Explaining and understanding PLs exponent is a large part of the theory of critical phenomena, in which many materials behave identically around phase transitions – a phenomenon physicists call "universality," and which is still partially understood. Power laws have proven useful to describe and understand networks. Biology has also many scaling regularities, e.g. the daily energy intake of an animal of mass M is proportional to the $M^{3/4}$. This regularity was explained (Brown and West 2000) via simple physical reasoning, that eschew the need to talk about the feathers and the hair of animals. Simpler and deeper principles underlie the regularities instead. The same holds for economic laws. Power laws give the hope of robust, detail-independent economic laws.

1 Theory: Forces that generate power laws

1.1 Proportional random growth

Getting a power law

To explain distributional PLs, a central mechanism is proportional random growth (Sornette 2001). The process was developed in economics by Champernowne (1953) and Simon (1955). Things are more tractable in continuous time, see Gabaix (1999).

Take the example of cities, in an economy with a constant number of cities and a fixed total population. When the system grows, the same reasoning applies after normalizations – S is the normalized size of a city, e.g. as a multiple of the median city population. Suppose that each city i has a population S_t^i, and between t and $t+1$, increases by a growth rate γ_{t+1}^i:

$$S_{t+1}^i = \gamma_{t+1}^i S_t^i, \tag{1}$$

and suppose that the γ_{t+1}^i are identically and independently distributed, with density $f(\gamma)$, at least in the upper tail. Call $G_t(x) = P\left(S_t^i > x\right)$, the counter-cumulative distribution function. The equation of motion of G is:

$$G_{t+1}(x) = P\left(S_{t+1}^i > x\right) = P\left(S_t^i > x/\gamma_{it}\right) = E\left[G_t\left(x/\gamma_{it}\right)\right]$$

hence:

$$G_{t+1}(S) = \int_0^\infty G_t\left(\frac{S}{\gamma}\right) f(\gamma)\, d\gamma.$$

Its steady state distribution G, if it exists, satisfies

$$G(S) = \int_0^\infty G\left(\frac{S}{\gamma}\right) f(\gamma)\, d\gamma. \tag{2}$$

One can try the functional form $G(S) = a/S^{\zeta}$, where a is a constant. Plugging it in (2) gives: $1 = \int_0^\infty \gamma^{\zeta} f(\gamma)\, d\gamma$, i.e.

$$E\left[\gamma^{\zeta}\right] = 1. \tag{3}$$

The steady state distribution is (in the upper tail) Pareto, with an exponent ζ that satisfies equation 3.

To make sure that the steady state distribution exists, one needs some friction, e.g. a force that prevents small cities from becoming too small.

Getting a Zipf's law

We see that proportional random growth leads to a PL. Why should the exponent $\zeta = 1$ appear in so many economic systems? An answer is the following (see Gabaix 1999 and Rossi-Hansberg and Wright 2007). Suppose that the random growth process (1) holds through most the distribution, and that the system has constant size. Then, $E[S_{t+1}] = E[\gamma] E[S_t]$. As the system has constant size, then we need $E[S_{t+1}] = E[S_t]$, hence $E[\gamma] = 1$. That means that $\zeta = 1$ is a solution of Eq. 3. In other terms, to get Zipf's law, we need a random growth processes with small frictions.

In sum, proportional random growth with frictions leads to PLs, and proportional random growth with small frictions leads to a special type of PL, Zipf's law.

1.2 Inheritance via algebraic transformation

Power laws have excellent inheritance and aggregation properties. The property of being distributed according to a PL is conserved under addition, multiplication, power transformation, min, and max. The general rule is that when we combine two PL variables, the fatter-tailed (i.e., the one with the smaller exponent) dominates. Call ζ_X the PL exponent of X, with $\zeta_X = +\infty$ if X is thinner than any PL, e.g. is a Gaussian. For X and Y independent random variables, and $\beta > 0$ a constant, we have: $\zeta_{X+Y} = \zeta_{X\cdot Y} = \zeta_{\max(X,Y)} = \min(\zeta_X, \zeta_Y)$, $\zeta_{\min(X,Y)} = \zeta_X + \zeta_Y$, $\zeta_{\alpha X} = \zeta_X$, $\zeta_{X^\alpha} = \zeta_X/\alpha$ (see Jessen and Mikosch 2006). Those properties generate new

PLs from old ones. For instance, if mutual funds are PL distributed, then many of their action (e.g., trading volumes, or the price movements they create) will be PL distributed (Gabaix 2006).

1.3 Equilibrium economic mechanisms

Optimization with PL objective function. The early example is the Allais-Baumol-Tobin model of demand for money (see also Mulligan and Shleifer 2005 and Gabaix et al. 2003). Costs and benefits are power functions of the variables of interest, so that maximization also yields a PL – there, money demand is a proportional to the interest rate to the power $-1/2$. PL in, PL out.

Matching talents in the upper tail. Another way to generate PLs is in matching the talent of individuals with a large firms or audiences, à la Rosen (1982). For instance, Gabaix and Landier (2006), study the market for executives. They derive that, in the upper tail, calling $T(x)$ the talent of an individual in the x upper quantile, then $T'(x)$ is approximately a power function x^{α} for all well-behaved distributions. As a result, the competitive matching process generates a PL relation between CEO pay and firm size, and a PL of the pay distribution. Huge differences in pay reward minuscule talent differences in talent. The PL form of T' is likely to be useful in other superstars markets.

2 Empirics: the main power laws of economics

2.1 Old macroeconomic scaling laws

The first quantitative law of economics is probably the quantity theory of money, which, not coincidentally, is a scaling relation. It states that the price level P is proportional to the mass of money in circulation M, divided by the gross domestic product Y, times a prefactor V: $P = VM/Y$. If the money supply doubles while GDP remains constant, prices double – a nice scaling law, relevant for policy.

More modern, we have the Kaldor's stylized facts on economic growth: with K the capital stock Y GDP, L population, r the interest rate, K/Y, wL/Y, and r, are roughly constant across time and countries. Explaining these facts led Solow to his growth model.

2.2 Reasonably old and well-established laws

Income and wealth. The first PL is the Pareto law of income or wealth, which states that the tail distribution of income (or, respectively, wealth), is PL. The tail exponent of income seems to vary between 1.5 and 3, while the tail exponent of wealth is more stable. While, starting with Champernowne (1953), many models have been proposed to it (mainly along the lines of random growth) though it is intriguingly unclear why the exponent is rather stable across economies.

Firm sizes. The bulk of the distribution of firm sizes is well described by a Zipf's law (Figure 1). This severely constrains models of firm growth, and means that idiosyncratic shocks of large firms may affect GDP (Gabaix 2006). Zipf's law holds for different measures of firm sizes and countries (Axtell 2001, Fujiwara *et al.* 2004, Gabaix and Landier 2006).

City sizes. In the upper tail, Zipf's law holds generally well across times and countries (Gabaix and Ioannides 2004).

Gibrat's law for the growth rate of cities is shown in the U.S. by Ioannides and Overman (2003).

Roberts' law for executive compensation. Across times and countries, an executive heading a firms of size S earns an amout proportionall to S^{κ}, for a κ around 1/3. Superstars models explain the presence of this scaling (Gabaix and Landier 2006), but the reason for the 1/3 value remains a mystery.

2.3 More recently proposed laws

Power law of stock market activity: returns, trading volume, and trading frequency. Following Mandelbrot, the following regularities have been found. Stock market returns (over 1 minute to 1 week) have PL tails, with an exponent around 3 (Gopikrishnan et al. 1999). Individual trades have a PL exponent around 1.5 (Gopikrishnan et al. 2000). The number of trades executed over a short horizon has an exponent close to 3 (Plerou et al. 2000). There is no consensus about the origins for those regularities. The fat-tailed of returns might come from GARCH effects. One view (Gabaix et al. 2003, 2006) attributes it to the trades of large institutional investors in relatively illiquid markets, which creates spikes in returns and volume, and generates empirically found exponents.

Supply of regulations. Mulligan and Shleifer (2005) establish another candidate law. In U.S. states, the quantity of regulation is a PL of population.

3 Estimation of power laws

How does one estimate a distributional PL? We take the example of n cities in the upper tail, ordering them by size, $S_{(1)} \geq \ldots \geq S_{(n)}$. One method is Hill's estimator:

$$\widehat{\zeta}^{Hill} = (n-1) \,/ \sum_{i=1}^{n-1} \left(\ln S_{(i)} - \ln S_{(n)}\right)$$

which has a standard error $\widehat{\zeta}^{Hill} n^{-1/2}$. The second method is a "log rank log size regression," where $\widehat{\zeta}$ the slope in the regression of the log rank i on the log size:

$$\ln(i - s) = \text{constant} - \widehat{\zeta}^{OLS} \ln S_{(i)} + \text{noise}$$

which has a standard error standard error is $\widehat{\zeta}^{OLS} \cdot (n/2)^{-1/2}$. s is a shift, $s = 0$ is typical, but $s = 1/2$ is optimal (Gabaix and Ibragimov 2006). Both methods have pitfalls, as true errors are often larger than nominal standard errors (Embrechts, et al. 1997, Gabaix and Ioannides 2004).

Xavier Gabaix

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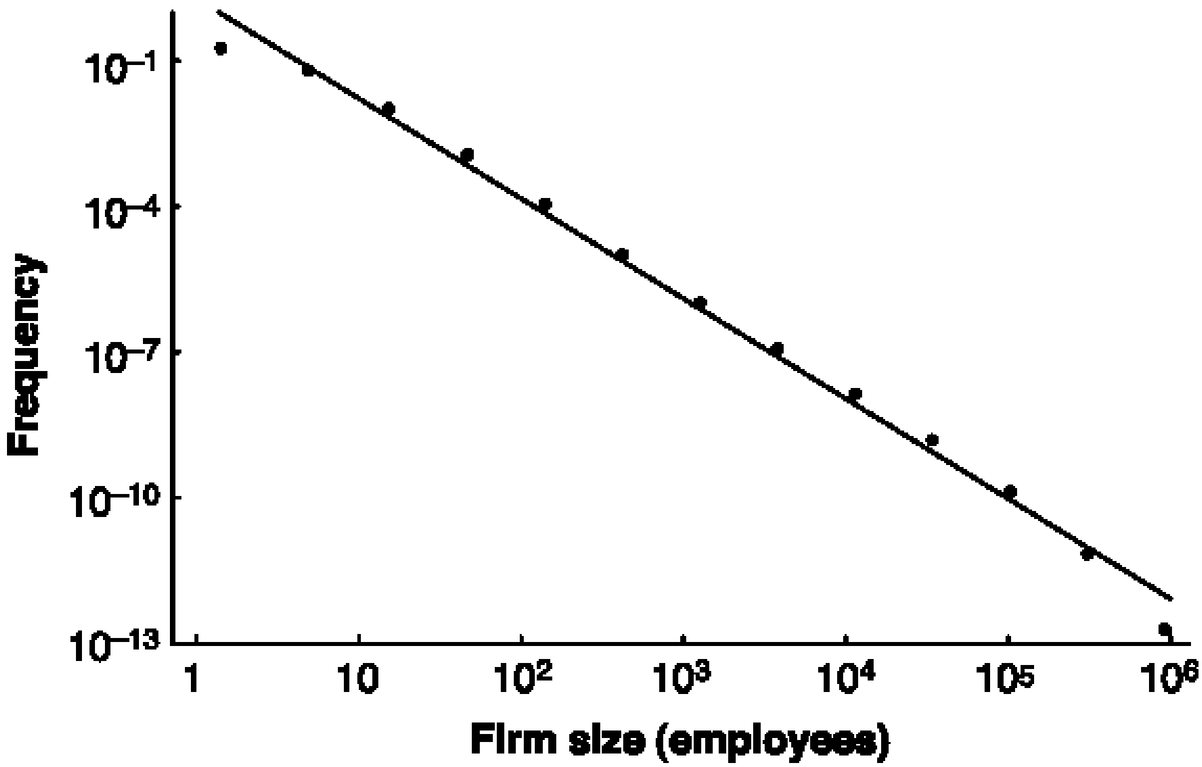
CAPTION FOR FIGURE 1:

Log frequency $\ln f(S)$ vs log size $\ln S$ of U.S. firm sizes for 1997. OLS fit gives a slope of $1+\zeta = 2.059$ (s.e.= 0.054; $R^2 = 0.992$). This corresponds to a frequency $f(S) = kS^{-2.059}$, i.e. a power law distribution with exponent $\zeta = 1.059$. Indeed, if $P(\text{Size} > S) = kS^{-\zeta}$, the density is $f(S) = k\zeta S^{-(\zeta+1)}$. This is very close to Zipf's law, which says that $\zeta = 1$. Source: Axtell (2001).

CROSS-REFERENCES TO OTHER ENTRIES: ARCH, Economics of Superstars, Econophysics, Firms, Income Inequality: Measurement, Quantity theory of Money, System of Cities, Wealth.



Frequency
10^{-1}
10^{-4}
10^{-7}
10^{-10}
10^{-13}
1
10
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
Firm size (employees)